and for the function S

$$S(\xi) = k_2^{-1}[\exp(\xi) - (1 + \xi + \tfrac{1}{2}\xi^2)] \tag{21}$$

where $\xi = k_1 u^+$.

An eddy viscosity model of the form $\epsilon^+ = k_1 S$ for the wall region has been proposed by Spalding.[4] Clearly, Spalding's formula as well as the model proposed by Kleinstein[2] coincide with the present model in the region where $\tau^+ = 1$.

It is apparent that once the asymptotic condition lim $(s/y^+) \to 1$ has been realized in Σ, as y^+ increases this condition continues to hold. An increase in y^+ however implies a farther penetration into the core region, but on the other hand as was shown in the core analysis the model $\epsilon^+ = k_1 y^+ \tau^+$ is valid for $\Sigma + C$. Thus we are led to the conclusion that for sufficiently large Reynolds numbers the eddy viscosity model $\epsilon^+ = k_1 S \tau^+$ holds through the entire pipe. It can be shown (Ref. 5) that "sufficiently large" means $0(\delta_\tau^{1/4} \ln \delta_\tau) \ll 1$.

III. Comparison with Experimental Results

The turbulent pipe flow problem as obtained by the aforementioned analysis is defined by the differential equation

$$du^+/dy^+ = \tau^+/(1 + \epsilon^+) \quad 0 < y^+ \leq \delta_\tau^{-1} \tag{22}$$

subject to the boundary conditions,

$$u^+ = 0 \quad \text{at} \quad y^+ = 0 \tag{22a}$$

where $\tau^+ = 1 - y^+\delta_\tau$, $\epsilon^+ = k_1 S \tau^+$, $S = k_2^{-1}[\exp(\xi) - (1 + \xi + \frac{1}{2}\xi^2)]$, $\xi = k_1 u^+$ provided $\lim(S/y^+) \to 1$ as $y^+ \to y_\Sigma{}^+$.

This theory is complete with the exception of the two arbitary constants k_1 and k_2 which must be determined experimentally. From Laufer's[6] experiment at $R_{\max} = (u_{\max}D/\nu) = 500{,}000$, the value of δ_τ as determined from pressure drop and $u_{\max}$ measurements was found to be 0.000114. Taking k_1—the von Kármán constant—as 0.4, the value of k_2 is determined in such a way that upon integrating Eq. (22) with $\delta_\tau = 0.000114$ the resulting Reynolds number $R_{\max} = (u_{\max}D/\nu) = 2u_{\max}{}^+\delta_\tau{}^{-1}$ is equal to 500,000. A calculation with $k_2 = 11.0$ gave $u_{\max}{}^+ = 28.69$ and $R_{\max} = 5.033 \times 10^5$. With $k_2 = 11.0$ as an accepted value a large number of calculations has been carried out for different values of the parameter δ_τ. In Fig. (1) viscous dissipation $w_\mu = \nu^2 u_\tau{}^{-4}(du/dy)^2 = (du^+/dy^+)^2$ and turbulent energy production $P_\tau = \epsilon^+ w_\mu$ as computed by the present analysis are compared with Laufer's[6] results at the 50,000 and 500,000 Reynolds numbers. In Fig. (2), the mean velocity distributions for ten different Reynolds numbers are compared with Nikuradse's data as presented in Goldstein.[7] As can be seen from the above figures the agreement with experimental data is good. Additional comparisons for the friction law and turbulent shear distributions could be found in Ref. (5).

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Condensation Augmented Velocity of a Supersonic Stream

P. M. Sherman*

The University of Michigan, Ann Arbor, Mich.

Over the years, condensation phenomena have been considered as they arose in turbines, in rocket nozzles and in wind tunnels.[1] When condensation occurs, the particles generated as well as the changed conditions of temperature, pressure, etc., result in problems such that, in general, efforts have been made to somehow prevent the condensation. More recently, however, condensation has been suggested as a means of producing particles for colloidal thrustors[2,3] and for EHD propulsion.[4] Presented here is another example of a possible advantage of condensation which has to do with the effect of condensation on velocity in supersonic flow. In previous studies very little attention has been paid to the effects of phase change on the velocity.

When heat is added to a supersonic stream in a constant area duct, the velocity of the stream must decrease.[5] It is simply shown here, however, that if heat is added to a supersonic stream in a diverging section of nozzle, the heat addition can yield an increase in nozzle exit velocity, over that obtained in the same nozzle but without the heat addition. Initially, when the heat is added to the supersonic stream, the velocity decreases. However, as the stream continues to expand, the increase in local temperature which results from the heat addition has a smaller effect on the velocity than the increase in what may be called "effective" stagnation temperature does. The velocity then increases above what it would be without the heat addition. Since nozzle exit velocity is a measure of specific impulse, increased nozzle exit velocity is of particular interest in propulsion schemes.

Adding heat to a supersonic stream without creating large disturbances ordinarily would present some problems. However, transferring the latent heat of vaporization to the stream by condensation can be an effective means of doing just that. One advantage of such heat transfer for a velocity increase is that it can be accomplished without any increase in the temperature which any solid boundary sees, since revaporization occurs when the flow is decelerated near the wall.

When homogeneous condensation occurs in a supersonic nozzle the particles have been shown to be so small[6] that it may be assumed they have the same velocity and temperature as the flowing gas. Then for steady-state one-dimensional flow of an ideal gas with condensation, the energy equation may be written as

$$C_pT + u^2/2 = C_pT_0 + gL \tag{1}$$

where C_p is the specific heat at constant pressure, (considered constant) T is local temperature, u is local velocity, T_0 is stagnation temperature upstream of the condensation, g is mass fraction of condensed vapor, L is latent heat. gL represents the heat added and $C_pT_0 + gL$ could be regarded as the "effective" stagnation temperature mentioned earlier. The velocity at any station is

$$u = \{2[C_pT_0 + gL - C_pT]\}^{1/2} \tag{2}$$

For "frozen" flow (without condensation) the velocity would be

$$u_n = \{2[C_pT_0 - C_pT_n]\}^{1/2} \tag{3}$$

where subscript n refers to the "frozen" flow. It can then be

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* Associate Professor, Department of Aerospace Engineering.

seen that when

$$(gL - C_pT) > (-C_pT_n) \quad \text{or} \quad gL > (C_pT - C_pT_n) \tag{4}$$

then u is greater than u_n for the same geometry and stagnation chamber conditions. The cross-over point where $u < u_n$ upstream and $u > u_n$ downstream then occurs when

$$T - T_n = gL/C_p \tag{5}$$

The simplest case of flow with condensation is the case for one-component equilibrium condensation. For this case the cross-over point where $u_n = u$ may be determined graphically by computing T and T_n as functions of nozzle area ratio and plotting $T - T_n$ vs gL/C_p. Unfortunately, even in this simplest case there is no direct solution for the cross-over point. For the equilibrium-condensation of a vapor in a carrier gas the solution is somewhat more complicated[7] but can be done in a similar manner. For nonequilibrium condensation the calculation is of course much more complicated and the sudden onset of condensation which occurs after supercooling will always produce a sharp decrease in velocity but further downstream conditions are not very far from those for equilibrium condensation.

An example of how the velocity changes for flow with condensation in a supersonic nozzle is shown in Fig. 1. The particular conditions used were chosen for an experimental program.[6] The solutions are for zinc vapor in a helium carrier gas expanded in a ~7.7° conical nozzle. Velocity is plotted vs distance downstream from the throat of the nozzle for both equilibrium condensation and nonequilibrium condensation with the isentropic "frozen" velocity shown for comparison. The nonequilibrium solution was based on the classical liquid drop model of nucleation.[8] $u - u_n$ is shown in Fig. 2 for equilibrium as well as nonequilibrium condensation. The mass fraction of condensate g as a function of distance downstream is also shown.

Although the velocity at the exit of the small nozzle considered here does not increase very much as a result of condensation, the results do show how velocity can be increased with no increase in the temperature at the walls. It can be seen that the increase in velocity becomes larger as the nozzle becomes longer. In the limit, of course, $u \rightarrow u_{\max}$ and the latent heat L is a measure of the maximum possible increase in the velocity since: $[u_{\max}{}^2 - u_{n\max}{}^2] \rightarrow 2\,gL$. Table I shows some values of $L^{1/2}$ in meters/sec for comparison purposes. It can be seen that zinc is not especially high on the list. L/C_pT_0 is also indicated in Table 1 since

$$u_{\max}/u_{n\max} \rightarrow [1 + gL/C_pT_0]^{1/2} \tag{6}$$

The values in the Table are for the pure substances. The temperatures correspond to saturation at one atmosphere pressure.

It should be pointed out that when a mixture of a gas and a vapor are employed, the increase in velocity which may be obtained by condensation will depend on the specific heat of the mixture rather than on the specific heat of the vapor. If an inert gas of low molecular weight (high specific heat) is employed with the condensing vapor, the vapor will condense more rapidly (after condensation has once started). The gas of low molecular weight tends to keep the temperature of the mixture lower after condensation but the fractional increase in velocity as a result of condensation will be less, since the velocity without condensation also is increased (if the molecular weight of the gas is less than the molecular weight of the vapor).

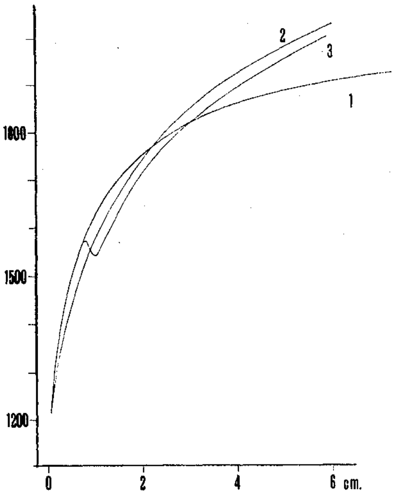

Fig. 1 Local velocity (m/sec) as a function of distance downstream from the throat of the nozzle (cm). 1) "Frozen" flow (without condensation); 2) flow with equilibrium condensation; 3) flow with nonequilibrium condensation. (Stagnation temperature—2500°K; stagnation pressure—2500 psia; mass fraction of zinc in helium—0.9.)

Table 1 Properties of some condensing substances (at one atmosphere pressure)

	T, °K	L, kcal/mole	$L^{1/2}$, m/sec	L/C_pT
Al	2720	20.2	3290	5.2
Li	1599	30.8	4330	3.9
Mg	1390	30.8	2300	4.2
Na	1156	24.6	2130	4.3
Zn	1181	27.6	1330	4.7
CO_2	194	6.03	756	6.3
H_2O	374	9.33	1475	7.1

One can consider separately the effect of change in molecular weight on velocity when a condensing heavier vapor is added to a lighter gas. Equation (6) then becomes

$$u_{\max}/u_{g\max} \rightarrow [C_p/C_{p_g} + \alpha gL/C_{p_v}T_0]^{1/2} \tag{7}$$

Where α is the ratio of molecular weight of the gas to the molecular weight of the vapor, C_{p_v} is the specific heat of the pure vapor, and the subscript g refers to the gas. If one used a metal vapor whose molecular weight was about twice that of the gas (such as, say, lithium in helium) with a mass fraction of about $\frac{1}{2}$, the velocity ratio [Eq. (7)] would approach

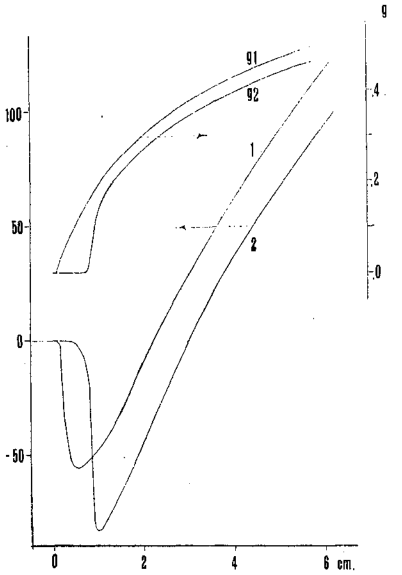

Fig. 2 Local difference in velocity (m/sec) between flow with and without condensation as a function of distance downstream of the nozzle throat (cm). 1) Equilibrium condensation; 2) nonequilibrium condensation. Curves g_1 and g_2 show the mass fraction of condensate for equilibrium and nonequilibrium condensation, respectively.

about 1.3. Of course, as α decreases, the resultant velocity ratio decreases. With the materials considered here, if α is decreased below about $\frac{1}{4}$, the velocity advantage begins to disappear.

In conclusion, when maintenance of moderate wall temperatures is required, it appears that condensation in a supersonic nozzle is a promising way of increasing exit velocity (specific impulse) for propulsion units.

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Population Inversions behind Normal Shock Waves

John D. Anderson Jr.* and Michael T. Madden†
U. S. Naval Ordnance Laboratory, White Oak, Md.

Introduction

As early as 1962, Basov and Oraevskii[1] suggested that population inversions in molecular systems could be created by rapid heating or cooling of the system. Subsequently, Hurle and Hertzberg[2] suggested that population inversions could be obtained in the rapid, nonequilibrium expansion of an initially hot gas through a supersonic nozzle. Indeed, experimental and theoretical results for such inversions, as well as laser stimulated emission, have been recently obtained in expansions of CO_2–N_2–He or H_2O mixtures (see the reviews in Refs. 3–5).

In contrast to rapid cooling, the present Note describes a theoretical study of vibrational population inversions created by rapid heating behind a normal shock wave in CO_2–N_2–He mixtures. On one hand, the creation of population inversions behind normal shock waves by means of shock-initiated chemical reactions have been studied by Oraevskii[6] and Gross et al.[7] Here, the inversions are formed directly by the preferential conversion of chemical energy into electronic[6] or vibrational[7] energy of the product species; this is the general principle of chemical lasers. On the other hand, the present Note examines the purely vibrational relaxation processes behind a normal shock wave in CO_2–N_2–He mixtures. The results of this analysis indicate that population inversions between the (200) and (001), and between the (04°0) and (001) vibrational energy levels of CO_2, can be created by molecular vibrational energy exchange only. Moreover, the present Note assesses the potential of this nonequilibrium flow as a possible laser medium.

Vibrational Model

The vibrational model assumed for the present study is shown in Fig. 1, which contains the pertinent low-lying vibrational energy levels of CO_2 and N_2. This model is described in detail in Ref. 8; in essence, the participating energy levels are grouped into modes I and II which are assumed in equilibrium within themselves, but not with each other. The net vibrational energies per unit mass, e_{vib_I} and $e_{vib_{II}}$, are the dependent nonequilibrium variables, which are assumed to relax according to the equations

$$\dot{w}_I = d(e_{vib})_I/dt = (1/\tau_I)[(e_{vib})_I^{eq} - (e_{vib})_I] \quad (1)$$

$$\dot{w}_{II} = d(e_{vib})_{II}/dt = (1/\tau_{II})[(e_{vib})_{II}^{eq} - (e_{vib})_{II}] \quad (2)$$

Here, τ_I and τ_{II} are the characteristic relaxation times for modes I and II, respectively; they are averages which depend on τ_a, τ_b, and τ_c for the mixture (see Fig. 1).

This vibrational model is a reasonable *approximation* for the detailed translation-vibration (T-V) and vibration-vibration (V-V) energy transfers within the CO_2–N_2–He mixture. Details of the approximations and their justification are as follows: 1) The relaxation (excitation) of mode I is assumed to be governed by the T-V transfer, τ_c. This is justified because the much slower V-V transfer, τ_a, has only a weak influence on mode I. For example, for a mixture of 60% He, 1.9% CO_2 and 38.1% N_2 at room temperature, the data of Ref. 9 show $\tau_a/\tau_c = 24$. Obviously, for the mixtures of interest here, τ_c is the prevailing relaxation time for the net excitation of mode I. 2) For the relaxation of mode II, both T-V and V-V energy exchanges are taken into account through the use of both τ_a and τ_b in the calculation of τ_{II} (see Ref. 8). With this value of τ_{II}, MacDonald[10] has kindly pointed out that Eq. (2) becomes more precise as T_{vib_I} approaches the gas translational temperature, T. For the case of present interest, this condition prevails; mode I relaxes very rapidly and is reasonably equilibrated with T before most of the mode II relaxation takes place. This behavior has been clearly established in both experimental and theoretical

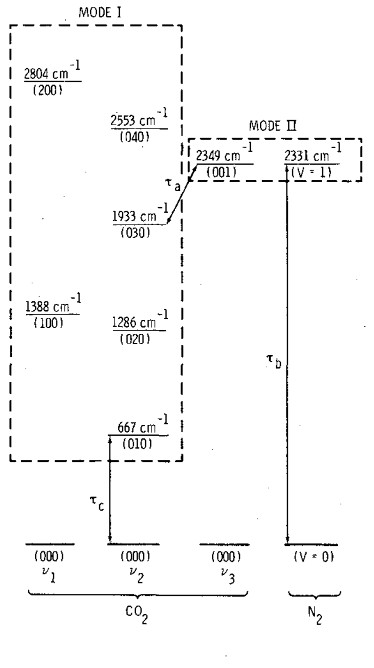

Fig. 1 Vibrational model.

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* Chief, Hypersonics Group, Aerophysics Division. Member AIAA.

† Aerospace Engineer, Aerophysics Division. Associate Member AIAA.